Analysis of a discretization method for the Richards' equation*

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Abstract

We analyse a discretization method for a class of degenerate parabolic problems that includes the Richards' equation. This analysis applies to the pressure-based formulation and considers both variably and fully saturated regimes. A regularization approach is combined with the Euler implicit scheme to achieve the time discretization. Equivalence between the two kinds of formulation is demonstrated for the semi-discrete case. Mixed finite elements are employed for the discretization in space. Error estimates are obtained, showing that the scheme is convergent.

Keywords: error estimates, Euler implicit scheme, mixed finite elements, regularization, degenerate parabolic problems, porous media, Richards' equation.
AMS classification: 65M12, 65M15, 65M60, 76S05, 35K65, 35K55

1 Introduction

A commonly accepted mathematical model of water flow in porous media is the Richards' equation, a nonlinear, possibly degenerate, parabolic differential equation. In the pressure formulation, Richards' equation [3] is expressed as

$$\partial_t \Theta(\psi) - \nabla \cdot K(\Theta)\nabla(\psi + z) = 0 \tag{1.1}$$

where ψ is the pressure head, Θ the saturation, K the conductivity and z the height against the gravitational direction. The equation (1.1) models the flow of a wetting fluid (water) in a porous media in the presence of a non-wetting fluid (air) supposed to be at constant pressure, 0. In the saturated region (where only water is present) we have $\psi \geq 0$, while $\psi < 0$ in the unsaturated domain. Different functional dependencies (retention curves) between ψ, K and Θ are proposed in the literature. Here we are interested in both partially saturated and saturated flow, therefore we retain the pressure ψ as primary unknown.

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As suggested in [1], applying the Kirchhoff transformation

$$\begin{array}{rcl} \mathcal{K} : \mathbb{R} & \longrightarrow & \mathbb{R} \\ \psi & \longmapsto & \displaystyle\int_0^{\psi} K(\Theta(s))\, ds \end{array} \tag{1.2}$$

leads to unknowns that are more regular. Since $K(\Theta(s))$ is positive, this transformation can be inverted and equation (1.1) can be rewritten in terms of a new variable, $u := \mathcal{K}(\psi)$. Defining now

$$\begin{array}{rcl} b(u) & := & \Theta \circ \mathcal{K}^{-1}(u), \\ k(b(u)) & := & K \circ \Theta \circ \mathcal{K}^{-1}(u), \end{array} \tag{1.3}$$

and letting e_z denote the vertical unit vector, equation (1.1) becomes

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\ e_z) = 0 \quad \text{in} \quad (0,T) \times \Omega. \tag{1.4}$$

Several papers are dealing with analysis and numerical methods for it. Euler methods are often employed for the discretization in time. Adaptive time stepping is studied, for example, in [19]. Iterative methods are considered for solving the resulting nonlinear equations in case of an implicit method (see, for example, [9], [5], and [7]). For the spatial discretization, mixed finite elements or finite volumes provide a good approximation of the solution ([10], [6]). Hybrid mixed finite elements are studied from an algorithmic point of view in [19].
However, most of the authors are mainly interested in computational aspects and less concerned with rigorous convergence results. With respect to this last aspect we mention recent papers like [20] (for the numerical analysis of a mixed finite element discretization), [6] (where convergence of an implicit finite volume method is proven by compactness arguments), [7] (for a relaxation scheme that applies to this equation too) and [15] (where error estimates are obtained for the unsaturated regime).
Here we consider a nondecreasing Lipschitz continuous nonlinearity b. Our numerical approach employs the lowest order Raviart–Thomas finite elements in space and Euler implicit in time, together with a regularization step. Specifically, with $N > 0$ integer, set $\tau = T/N$ and let $\mathcal{T}_h$ being a decomposition of Ω into closed d-simplices; h stands for the mesh-size. Then the numerical scheme under consideration reads

$$\begin{array}{l} b_\epsilon(p_h^n) + \tau \nabla q_h^n = b_\epsilon(p_h^{n-1}), \\ q_h^n + \nabla p_h^n + k(b(p_h^n)) e_z = 0, \end{array}$$

for $n = \overline{1, N}$; p_h^0 approximates u^0 in the finite dimensional approximation space. Here b_ϵ is a regular approximation of b depending on the small parameter $\epsilon > 0$. By p_h^n we denote a piecewise constant approximation of u and q_h^n is a Raviart-Thomas (RT_0) approximation of the flux -$(\nabla u + k(b(u))e_z)$, based on $\mathcal{T}_h$, both at $t = n\tau$.
As suggested in [2], to overcome the difficulties posed by the lack in regularity, equation (1.4) is first integrated in time. For the resulting problem a mixed variational formulation is stated.
Convergence is shown by obtaining first error estimates for the time discrete scheme, by following the ideas in [13]. Next, using the procedure described in [2], error estimates for the fully discrete scheme are obtained. In this setting, the equivalence between the two different formulations becomes essential since in this way results obtained for the conformal method can be transferred to the mixed one and vice-versa. The results are given here without proofs, which can be found in [17].

1.1 Notation and assumptions

In what follows Ω is a domain in $\mathbb{R}^d$ (with $d = 1, 2$ or 3). Let $J = (0, T]$ be a finite time interval. We are interested in solving equation (1.4) endowed with initial and boundary conditions,

$$\begin{array}{rcll} \partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))e_z) & = & 0 & \text{in} \quad J \times \Omega, \\ u & = & u^0 & \text{in} \quad 0 \times \Omega, \\ u & = & 0 & \text{on} \quad J \times \Gamma. \end{array} \tag{1.5}$$

Throughout this paper we make use of the following assumptions:

(A1) $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz continuous boundary.

(A2) $b \in C^1$ is non-decreasing and Lipschitz continuous.

(A3) $k(b(z))$ is continuous and bounded in z and satisfies, for all $z_1, z_2 \in \mathbb{R}$,

$$| k(b(z_2)) - k(b(z_1)) |^2 \leq C_k(b(z_2) - b(z_1))(z_2 - z_1).$$

(A4) $b(u_0)$ is essentially bounded (by 0 and 1) in Ω and $u_0 \in L^2(\Omega)$.

Assuming k bounded is not unrealistic since, for Richards' equation, k models the medium conductivity of the medium. The growth condition on $k(b(\cdot))$ (see also [15] and [16]) relaxes the more often assumed Lipschitz continuity of k (see, for example, [13] or [2]). Source terms can also be considered here, provided that they satisfy a similar growth condition as $k(b(u))$.
Here and below $(\cdot, \cdot)$ stands for the inner product on $L^2(\Omega)$ or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|$ for the norm in $L^2(\Omega)$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ for the norms in $H^1(\Omega)$, respectively $H^{-1}(\Omega)$. We use analogous notations for the inner product and the corresponding norm on $L^2(0, T; \mathcal{H})$, with $\mathcal{H}$ being either $L^2(\Omega)$, $H^1(\Omega)$, or $H^{-1}(\Omega)$. In addition, we often write u or $u(t)$ instead of $u(t, x)$ and use C to denote a generic positive constant, not depending on the discretization or regularization parameters.
Because of the degenerate character, one can only expect weak solutions for problem (1.5). Existence, uniqueness and essential bounds for a weak solution is studied in several papers (see, for example, [1], [14], and the references therein). Following [2] or [20] we integrate (1.5) in time and obtain, for every $t \in J$,

$$b(u(t)) + \nabla \cdot \int_0^t \vec{q}\,(s)\,ds = b(u^0) \tag{1.6}$$

in L^2 sense. From [2], the flux $\vec{q} := -\left(\nabla u + k(b(u))e_z\right)$ satisfies

$$\int_0^t \vec{q}\,d\tau \;\in\; H^1(J; (L^2(\Omega))^d) \cap L^2(J; (H^1(\Omega))^d) =: X. \tag{1.7}$$

2 Equivalent formulations

The spatial discretization is provided by mixed finite elements. The convergence proof makes use of the equivalence between the conformal formulation and the mixed one.

2.1 *The continuous case*

Integrated in time, problem (1.5) becomes
Problem 1. Find $u \in L^2(J, H_0^1(\Omega))$ such that $b(u) \in L^\infty(J \times \Omega)$, and for all $t \in J$ and $\phi \in H_0^1(\Omega)$ it holds

$$(b(u(t)) - b(u^0), \phi) + \int_0^t (\nabla u(s) + k(b(u(s)))e_z, \nabla\phi)ds = 0. \tag{2.1}$$

Here $b(u)$ models the water content, hence it is natural to assume it bounded almost everywhere in $J \times \Omega$. Moreover, $u \in L^2(0, T; H_0^1(\Omega))$ yields $b(u) \in L^2(0, T; H_0^1(\Omega))$ due to the Lipschitz continuity of b. Since $b(u) \in H^1(0, T; H^{-1}(\Omega))$ we have $b(u) \in C(0, T; L^2(\Omega))$ (see [12], chapter I), allowing a simplified mixed variational formulation.
A mixed formulation for Problem (1.5) reads
Problem 2. Find $(p, \tilde{q}) \in L^2(J \times \Omega)) \times X$ such that $b(p) \in L^\infty(J \times \Omega)$ and for all $t \in J$ the equations

$$\begin{aligned} (b(p(t)) - b(p^0), w) + (\nabla\tilde{q}(t), w) &= 0, & (2.2)\\ (\tilde{q}(t), v) - \int_0^t (p(s), \nabla v)ds + \int_0^t (k(b(p(s)))e_z, v)ds &= 0, & (2.3)\end{aligned}$$

hold for all $w \in L^2(\Omega)$ and $v \in H(div, \Omega)$, with $p^0 = u^0 \in L^2(\Omega)$.
The two problems are equivalent, as stated below.

Proposition 2.1 *$u \in L^2(J, H_0^1(\Omega))$ solves Problem 1 iff $(p, \tilde{q}) \in L^2(J \times \Omega)) \times X$ defined as*

$$(p, \tilde{q}) = (u, -\int_0^t (\nabla u(s) + k(b(u(s)))e_z)ds) \tag{2.4}$$

solves Problem 2. Moreover, in this case we have $p \in L^2(J, H_0^1(\Omega))$.

2.2 *The semi-discrete case*

As mentioned in the introduction, difficulties due to degeneracy can be overcomed by perturbing the original equation to a regular parabolic one. Such a technique has been successfully applied in the analysis of degenerate problems, and also allows developing effective numerical schemes (see, for example, [13], [7], or [16]). Here we approximate b by b_ϵ, where $\epsilon > 0$ is a small perturbation parameter. A possible choice reads

$$b_\epsilon(u) = b(u) + \epsilon u. \tag{2.5}$$

b_ϵ has the same properties as b but its derivative is bounded from below by ϵ.
With $N > 1$ being an integer giving the time step $\tau = T/N$ and $t_n = n\tau$, the regularized semi-discrete conformal problem reads
Problem 3. Let $n = \overline{1, N}$ and u^{n-1} be given. Find $u^n \in H_0^1(\Omega)$ such that, for all $\phi \in H_0^1(\Omega)$,

$$(b_\epsilon(u^n) - b_\epsilon(u^{n-1}), \phi) + \tau(\nabla u^n + k(b(u^n))e_z, \nabla\phi) = 0. \tag{2.6}$$

Its mixed time discrete counterpart becomes
Problem 4. Let $n = \overline{1, N}$ and p^{n-1} given. Find $(p^n, q^n) \in L^2(\Omega) \times H(div, \Omega)$ such that

$$\begin{aligned} (b_\epsilon(p^n) - b_\epsilon(p^{n-1}), w) + \tau(\nabla q^n, w) &= 0, & (2.7)\\ (q^n, v) - (p^n, \nabla v) + (k(b(p^n))e_z, v) &= 0, & (2.8)\end{aligned}$$

for all $w \in L^2(\Omega)$, respectively $v \in H(div, \Omega)$, with $p^0 = u^0 \in L^2(\Omega)$.
As in the continuous case, the two problems above are equivalent.

Proposition 2.2 *Let $n = \overline{1,N}$ be fixed and assume $u^{n-1} = p^{n-1}$. Then $u^n \in H_0^1(\Omega)$ solves Problem 3 iff $(p^n, q^n) \in L^2(\Omega) \times H(div, \Omega)$ defined as*

$$(p^n, q^n) = (u^n, -(\nabla u^n + k(b(u^n))e_z)) \tag{2.9}$$

solve Problem 4. Moreover, we have $p^n \in H_0^1(\Omega)$.

3 Estimates for the time discrete case

Due to the equivalences proven above, stability and error estimates for the time discrete mixed formulation can be obtained by analyzing the Euler implicit scheme applied to Problem 3. Such results are obtained in [13] and can be transferred straightforwardly to the mixed formulation.

3.1 ***Stability***

An immediate consequence of Lemmas 4 and 5 in [13] is

Proposition 3.1 *Assuming (A1) - (A4), if, for any $n = \overline{1,N}$, (p^n, q^n) solve Problem 4, we have*

$$\tau \sum_{n=1}^{N} \|p^n\|_1^2 + \tau \sum_{n=1}^{N} \|q^n\|^2 \leq C. \tag{3.1}$$

Defining an initial flux q^0 in $[L^2(\Omega)]^d$ (see [17] for details) these estimates can be completed by

Proposition 3.2 *Assuming (A1) - (A4), if, for all $n = \overline{1,N}$, (p^n, q^n) solve Problem 4, for any $k > 0$ we have*

$$\begin{array}{c} \sum_{n=1}^{k} (b_\epsilon(p^n) - b_\epsilon(p^{n-1}), p^n - p^{n-1}) + \tau\|q^k\|^2 + \tau \sum_{n=1}^{k} \|q^n - q^{n-1}\|^2 \leq C\tau, \\ \sum_{n=1}^{N} \tau \|\nabla \cdot q^n\|^2 \leq C. \end{array} \tag{3.2}$$

3.2 ***Error estimates***

We use the notations

$$\begin{aligned} \overline{u}^n &= \tfrac{1}{\tau} \int_{t_{n-1}}^{t_n} u(t)dt, \\ p_\Delta(t) &= p^n, \quad \text{for} \quad t \in (t_{n-1}, t_n], \\ e_b(u) &= b(u) - b_\epsilon(p_\Delta), \end{aligned} \tag{3.3}$$

where $n = \overline{1,N}$ and $\overline{u}^0 = u^0$.
For the semi-discrete mixed discretization scheme we obtain the following

Theorem 3.1 *Assuming (A1) - (A4), if if u is the weak solution of Problem 1 and (p^n, q^n) solve Problem 4 ($n = \overline{1,N}$), we get*

$$\begin{aligned} & \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (b_\epsilon(u(t)) - b_\epsilon(p^n), u(t) - p^n)dt \\ + \quad & \| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (u(t) - p^n)dt\|_1^2 + \|\tilde{q}(T) - \tau \sum_{n=1}^{N} q^n\|^2 \\ \leq \quad & C(\tau + \epsilon). \end{aligned} \tag{3.4}$$

Remark 3.1 *Since b_ϵ is a perturbation of order ϵ for b we can replace the scalar product in (3.4) by $\int_0^T (b(u(t)) - b(p_\Delta(t)), u(t) - p_\Delta(t))dt$. This immediately implies an error estimate for the saturation,*

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|b(u(t)) - b(p^n)\|^2 dt \leq C(\tau + \epsilon).$$

Remark 3.2 *Applying the backward Euler method to degenerate parabolic problems that can be seen as semigroups generated by sub-gradients in Hilbert spaces leads to optimal estimates (see [18]). This is not always our case, particularly because we have to deal with a nonlinear convection term. However, if the problem to be solved numerically is so that Rulla's result hold, optimal estimates can be obtained also for the present scheme.*

4 Estimates for the full discretization

For the spatial discretization we let $\mathcal{T}_h$ be a regular decomposition of $\Omega \subset \mathbb{R}^d$ into closed d-simplices; h stands for the mesh-size. To avoid technicalities, Ω is assumed polygonal, satisfying $\overline{\Omega} = \cup_{T \in \mathcal{T}_h} T$.
The discrete subspaces $W_h \times V_h \subset L^2(\Omega) \times H(div, \Omega)$ are defined as

$$\begin{aligned} W_h &:= \{p \in L^2(\Omega) |\ p \text{ is constant on each element } T \in \mathcal{T}_h\}, \\ V_h &:= \{\overrightarrow{q} \in H(div, \Omega) |\ \overrightarrow{q}_{|T} = \overrightarrow{a} + b\overrightarrow{x} \quad \text{for all } T \in \mathcal{T}_h\}. \end{aligned} \tag{4.1}$$

So W_h denotes the space of piecewise constant functions, while V_h is the RT_0 space (see [4]). Further we make use of the usual L^2 projector

$$P_h : L^2(\Omega) \to W_h, \quad ((P_h w - w), w_h) = 0 \qquad \forall w_h \in W_h. \tag{4.2}$$

Taking a space $\widetilde{V}$ slightly better than $H(div, \Omega)$ (for example, $H(div, \Omega) \cap (L^s(\Omega))^d$ with an $s > 2$), a projector Π_h can be defined as (see [4], p.131)

$$\Pi_h : \widetilde{V} \to V_h, \quad (\nabla \cdot (\Pi_h v - v), w_h) = 0 \tag{4.3}$$

for all $w_h \in W_h$. With $r \geq 0$, for the operators defined above we have

$$\begin{aligned} \|w - P_h w\| &\leq Ch^r \|w\|_r, \\ \|v - \Pi_h v\| &\leq Ch^r \|v\|_r, \end{aligned} \tag{4.4}$$

for any $w \in H^r(\Omega)$ and $v \in (H^r(\Omega))^d$.
Before proceeding with the fully discrete approximation scheme we rewrite Problem 4 (continuous in space) as

Problem 5. Let $n = \overline{1, N}$. Find $(p^n, q^n) \in L^2(\Omega) \times H(div, \Omega)$ such that

$$(b_\epsilon(p^n), w) - (b_\epsilon(p^0), w) + \tau(\sum_{j=1}^{n} \nabla q^j, w) = 0, \tag{4.5}$$

$$(q^n, v) - (p^n, \nabla v) + (k(b(p^n))e_z, v) = 0, \tag{4.6}$$

for all $w \in L^2(\Omega)$ and $v \in H(div, \Omega)$, with $p^0 = u^0$.
The fully discrete mixed finite element approximation reads

Problem 6. Let $n = \overline{1, N}$. Find $(p_h^n, q_h^n) \in W_h \times V_h$ such that

$$(b_\epsilon(p_h^n), w_h) + \tau(\sum_{j=1}^{n} \nabla q_h^j, w_h) = (b_\epsilon(p_h^0), w_h), \tag{4.7}$$

$$(q_h^n, v_h) - (p_h^n, \nabla v_h) + (k(b(p_h^n))e_z, v_h) = 0, \tag{4.8}$$

for all $w_h \in W_h$ and $v_h \in V_h$.

Initially we take $p_h^0 = b_\epsilon^{-1}(P_h b_\epsilon(u^0))$. Since $P_h b_\epsilon(u^0)$ is constant on any $T \in \mathcal{T}_h$, the same holds for $b_\epsilon^{-1}(P_h b_\epsilon(u^0))$, so $p_h^0 \in W_h$. Moreover, with this choice, for all $w_h \in W_h$, we obtain

$$(b_\epsilon(p_h^0), w_h) = (b_\epsilon(u^0), w_h) = (b_\epsilon(p^0), w_h).$$

We start with a-priori estimates for the fully discrete case.

Proposition 4.1 *Assuming (A1) - (A4), if* (p_h^n, q_h^n) *solve Problem 6* ($n = \overline{1, N}$)*, we have*

$$\begin{aligned} \|p_h^n\|^2 + \|q_h^n\|^2 &\leq C, \\ (b_\epsilon(p_h^n) - b_\epsilon(p_h^{n-1}), p_h^n - p_h^{n-1}) &\leq C\tau. \end{aligned} \tag{4.9}$$

Applying now techniques developed in [2] we estimate the errors induced by the spatial discretization.

Theorem 4.1 *Assuming (A1)-(A4), if* $(p^n, q^n) \in L^2(\Omega) \times H(div, \Omega)$, $(p_h^n, q_h^n) \in W_h \times V_h$ *solve, for* $n = \overline{1, N}$*, Problems 5 and 6, we obtain*

$$\begin{aligned} & \sum_{n=1}^N (b_\epsilon(p^n) - b_\epsilon(p_h^n), p^n - p_h^n) + \tau \sum_{n=1}^N \|\Pi_h q^n - q_h^n\|^2 \\ + \quad & \tau \| \sum_{n=1}^N (q^n - q_h^n)\|^2 + \tau \| \sum_{n=1}^N (p^n - p_h^n)\|^2 \\ \leq \quad & C\left(\frac{h^2}{\tau} + \sum_{n=1}^N \|q^n - \Pi_h q^n\|^2\right). \end{aligned} \tag{4.10}$$

Combining the estimates in Theorems 3.1 and 4.1 we get, for the fully discrete scheme

Theorem 4.2 *Assuming (A1)-(A4), we get*

$$\begin{aligned} & \| \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (u(t) - p_h^n) dt\|^2 + \|\tilde{q}(T) - \tau \sum_{n=1}^N q_h^n\|^2 \\ \leq \quad & C(\tau + \epsilon + h^2 + \tau \sum_{n=1}^N \|q^n - \Pi_h q^n\|^2). \end{aligned} \tag{4.11}$$

Corollary 4.1 *As in Theorem 4.2, for the scalar product we have*

$$\begin{aligned} & \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (b_\epsilon(u(t)) - b_\epsilon(p_h^n), u(t) - p_h^n) dt \\ & \leq C\left(\tau^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} + h^2/\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}} \sum_{n=1}^N \|q^n - \Pi_h q^n\|^2\right). \end{aligned} \tag{4.12}$$

Assuming additionally

(A5) $q^n \in H^1(\Omega)^d$ for all $n = \overline{1, N}$,

recalling (4.4), the estimates in Theorem 4.2 and Corollary 4.1 become

$$\begin{aligned} & \| \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (u(t) - p_h^n) dt\|^2 + \|\tilde{q}(T) - \tau \sum_{n=1}^N q_h^n\|^2 \leq C(\tau + \epsilon + h^2), \\ & \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (b_\epsilon(u(t)) - b_\epsilon(p_h^n), u(t) - p_h^n) dt \leq C\left(\tau^{\frac{1}{2}} + \epsilon^{\frac{1}{2}} + h^2/\tau^{\frac{1}{2}}\right). \end{aligned} \tag{4.13}$$

Remark 4.1 *Obviously (A5) is fulfilled in one spatial dimension, since then* $H(div, \Omega)$ *and* $H^1(\Omega)$ *coincide. Assumption (A5) holds also in the multi-dimensional case provided* $\partial\Omega$ *is smooth enough and* k *is derivable. Then, following [11] (Chapter 4, Theorems 5.1 and 5.2), for any* $n = \overline{1, N}$, u^n *solving Problem 3 is in* $H^2(\Omega)$ *and the corresponding norm is bounded uniformly in* n *by a constant that, nevertheless, may depend on* τ*. Therefore* $q^n \in H^1(\Omega)$ *for all* $n \geq 1$ *and* $\|q^n\|_1 \leq C(\tau)$.

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